FROM FINITE TO INFINITE DIMENSION

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1. The Large Deviation Principle in Infinite-Dimensional Spaces

When establishing the Large Deviation Principle (LDP) for random vectors in infinite-dimensional spaces, a common strategy is to first establish the LDP for projections onto finite-dimensional subspaces and then lift these results to the original space. This approach is codified in the following theorem.

Theorem 1.1. *Suppose that $\{\mu_\varepsilon\}$ satisfies a weak LDP with a convex rate function I, and let $\mathcal{X}$ be a locally convex Hausdorff topological vector space. Assume that for each $\lambda \in \mathcal{X}^*$, the limits $\Lambda_\lambda(t) = \lim_{\varepsilon\to 0} \varepsilon\Lambda_{\mu_\varepsilon}(t\lambda/\varepsilon)$ exist as extended real numbers, and that $\Lambda_{\lambda(t)}$ is a lower semicontinuous function of $t \in \mathbb{R}$. If for every $\lambda \in \mathcal{X}^*$ and every $a \in \mathbb{R}$,*

$$\inf_{x:\langle\lambda,x\rangle>a} I(x) \le \inf_{z>a} \Lambda^*_\lambda(z),$$

then $I = \Lambda^*$, and consequently, Λ^* controls a weak LDP associated with $\{\mu_\varepsilon\}$.

The ingredients for the proof are summarized in the following subsections.

1.1. Varadhan's integral lemma.

In this subsection we assume $\mathcal{X}$ to be a regular topological space.

Theorem 1.2 (Varadhan)**.** *Suppose that $\{\mu_\varepsilon\}$ satisfies the LDP with a good rate function $I : \mathcal{X} \to [0,\infty]$, and let $\varphi : \mathcal{X} \to \mathbb{R}$ be any continuous function. Assume further that either the tail condition*

$$\lim_{M\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{E}\Big[e^{\frac{\varphi(Z_\varepsilon)}{\varepsilon}} \mathbb{1}_{\{\varphi(Z_\varepsilon)\ge M\}}\Big] = -\infty \tag{1.1}$$

or the following moment condition for some $\gamma > 1$ holds:

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{E}\Big[e^{\frac{\gamma\varphi(Z_\varepsilon)}{\varepsilon}}\Big] < \infty. \tag{1.2}$$

Then,

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}\Big[e^{\frac{\varphi(Z_\varepsilon)}{\varepsilon}}\Big] = \sup_{x\in\mathcal{X}} [\varphi(x) - I(x)].$$

The theorem is a consequence of the following three lemmas.

Lemma 1.3. *If $\varphi : \mathcal{X} \to \mathbb{R}$ is lower semicontinuous and the large deviations lower bound holds with $I : \mathcal{X} \to [0, \infty]$, then*

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{E}\left[\exp\left(\frac{\varphi(Z_\varepsilon)}{\varepsilon}\right)\right] \geq \sup_{x \in \mathcal{X}} [\varphi(x) - I(x)].$$

Lemma 1.4. *If $\varphi : \mathcal{X} \to \mathbb{R}$ is an upper semicontinuous function for which the tail condition (1.1) holds, and the large deviations upper bound holds with the good rate function $I : \mathcal{X} \to [0, \infty]$, then*

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{E}\left[\exp\left(\frac{\varphi(Z_\varepsilon)}{\varepsilon}\right)\right] \leq \sup_{x \in \mathcal{X}} [\varphi(x) - I(x)].$$

Lemma 1.5. *Condition (1.2) implies the tail condition (1.1).*

Proof of Lemma 1.3. Fix $x \in \mathcal{X}$. Since φ is lower semicontinuous, for every $\delta > 0$, there exists an open neighborhood G of x such that

$$-\varphi(x) \geq -\inf_{y \in G} \varphi(y) - \delta.$$

Using Markov's inequality and the lower bound of the LDP, we have

$$\begin{aligned}
&\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{E}\Big(e^{\frac{\varphi(Z_\varepsilon)}{\varepsilon}}\Big) \geq \liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{E}\Big(e^{\frac{\varphi(Z_\varepsilon)}{\varepsilon}} \mathbb{1}_{Z_\varepsilon \in G}\Big) \\
&\geq \inf_{y \in G} \varphi(y) + \liminf_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(G) \\
&\geq \inf_{y \in G} \varphi(y) - \inf_{y \in G} I(y) \geq \varphi(x) - I(x) - \delta,
\end{aligned}$$

which proves the inequality since x and δ are arbitrary. □

Proof of Lemma 1.4. Note that by the semicontinuity of φ and I, and the regularity of $\mathcal{X}$, for each $x \in \mathcal{X}$ and each $\delta > 0$, there exists an open neighborhood A_x such that

$$\inf_{y \in \overline{A}_x} I(y) \geq I(x) - \delta \quad \text{and} \quad \sup_{y \in A_x} \varphi(y) \leq \varphi(x) + \delta,$$

which yields the local estimate $\mathbb{E}\Big(e^{\frac{\varphi(Z_\varepsilon)}{\varepsilon}} \mathbb{1}_{\{Z_\varepsilon \in A_x\}}\Big) \leq e^{\frac{\varphi(x)+\delta}{\varepsilon}} \mu_\varepsilon(A_x)$. For each $\alpha < \infty$, let $\{A_{x_i}\}_{i=1}^N$ be a finite cover of the compact set $\Psi_I(\alpha)$. Then,

$$\begin{aligned}
&\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{E}\Big[e^{\frac{\varphi(Z_\varepsilon)}{\varepsilon}} \mathbb{1}_{\{\varphi(Z_\varepsilon) \leq M\}}\Big] \\
&\leq \limsup_{\varepsilon \to 0} \max\Big\{M - \alpha, \max_{1 \leq i \leq N} \Big[\varphi(x_i) + \delta + \varepsilon \log \mu_\varepsilon\Big(A_{x_i}\Big)\Big]\Big\} \\
&\leq \max\left\{M - \alpha, \max_{1 \leq i \leq N} \left[\varphi(x_i) + \delta - \inf_{x \in \overline{A}_{x_i}} I(x)\right]\right\} \\
&\leq \max\Big\{M - \alpha, \max_{x \in \Psi_I(\alpha)} [\varphi(x) - I(x) + 2\delta]\Big\}.
\end{aligned}$$

By letting $\delta \to 0$ and then $\alpha \to \infty$, we obtain

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{E}\Big[e^{\frac{\varphi(Z_\varepsilon)}{\varepsilon}} \mathbb{1}_{\{\varphi(Z_\varepsilon) \leq M\}}\Big] \leq \sup_{x \in \mathcal{X}} [\varphi(x) - I(x)].$$

The proof is completed by using assumption (1.1), which implies

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{E}\Big(e^{\frac{\varphi(Z_\varepsilon)}{\varepsilon}}\Big) \\
&= \lim_{M\to\infty} \limsup_{\varepsilon\to 0} \max\Big\{\varepsilon \log \mathbb{E}\Big[e^{\frac{\varphi(Z_\varepsilon)}{\varepsilon}} \mathbb{1}_{\{\varphi(Z_\varepsilon)\le M\}}\Big], \varepsilon \log \mathbb{E}\Big[e^{\frac{\varphi(Z_\varepsilon)}{\varepsilon}} \mathbb{1}_{\{\varphi(Z_\varepsilon)> M\}}\Big]\Big\} \\
&= \limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{E}\Big[e^{\frac{\varphi(Z_\varepsilon)}{\varepsilon}} \mathbb{1}_{\{\varphi(Z_\varepsilon)\le M\}}\Big].
\end{aligned}$$

□

Proof of Lemma 1.5. The lemma follows from the observation that for any non-negative random variable X and any $\gamma > 1$,

$$\mathbb{E}\Big[X^\gamma \mathbb{1}_{\{X\ge\alpha\}}\Big] = \alpha^\gamma \mathbb{P}(X \ge \alpha) + \int_\alpha^\infty x^{\gamma-1} \mathbb{P}(X \ge x)\, dx.$$

Hence, $\mathbb{E}\Big[X^\gamma \mathbb{1}_{\{X\ge\alpha\}}\Big] \ge \alpha^{1-\gamma} \mathbb{E}\Big[X \mathbb{1}_{\{X\ge\alpha\}}\Big]$, which proves the lemma upon setting $X = \frac{\varphi(Z_\varepsilon)}{\varepsilon}$ and $\alpha = \frac{M}{\varepsilon}$. □

1.2. **Fenchel-Legendre transform.**

Theorem 1.6. *Let $\mathcal{X}$ be a locally convex Hausdorff topological vector space. Assume that μ_ε satisfies the LDP with a good rate function I. Suppose in addition that*

$$\overline{\Lambda}(\lambda) = \limsup_{\varepsilon\to 0} \varepsilon \Lambda_{\mu_\varepsilon}\left(\frac{\lambda}{\varepsilon}\right), \quad \forall \lambda \in \mathcal{X}^*. \tag{1.3}$$

(1) For each $\lambda \in \mathcal{X}^*$, $\lim_{\varepsilon\to 0} \varepsilon \Lambda_{\mu_\varepsilon}\left(\frac{\lambda}{\varepsilon}\right)$ exists, is finite, and

$$\Lambda(\lambda) = \sup_{x\in\mathcal{X}} [\langle \lambda, x\rangle - I(x)].$$

(2) If I is convex, then it is the Fenchel–Legendre transform of Λ, namely,

$$I(x) = \Lambda^*(x) := \sup_{\lambda\in\mathcal{X}^*} [\langle \lambda, x\rangle - \Lambda(x)].$$

(3) If I is not convex, then Λ^* is the affine regularization of I, i.e.,

$$\Lambda^* = \sup\{f : f \le I, f \text{ is convex}\}.$$

Proof. (1) Fix $\lambda \in \mathcal{X}^*$ and $\gamma > 1$. By assumption, Varadhan's lemma (Theorem 1.2) asserts that

$$\lim_{\varepsilon\to 0} \varepsilon \Lambda_{\mu_\varepsilon}\left(\frac{\lambda}{\varepsilon}\right) = \sup_{x\in\mathcal{X}} [\langle \lambda, x\rangle - I(x)] < \infty.$$

Since Λ is convex and $\Lambda(0) = 0$, it follows that $\Lambda > -\infty$.

(2) This follows from the duality lemma (Lemma D.1).

(3) From part (1), we have

$$\Lambda^*(x) = I^{**}(x) = \sup_{\lambda\in\mathcal{X}^*} \inf_{y\in\mathcal{X}} [\langle \lambda, x - y\rangle + I(y)] \le I(x).$$

By the Hahn-Banach separation theorem,

$$\sup\{f : f \le I, f \text{ is convex}\} = \sup\{\lambda + c : \lambda + c \le I, \lambda \in \mathcal{X}^*, c \in \mathbb{R}\}.$$

If $I \ge \lambda + c$, it is necessarily the case that $\inf_{y\in\mathcal{X}} [I(y) - \langle \lambda, y\rangle] \ge c$, or equivalently,

$$\inf_{y\in\mathcal{X}}[\langle\lambda, x-y\rangle + I(y)] \geq \langle\lambda, x\rangle + c,$$

from which the claim follows. □

1.3. **Proof of the theorem.**

Proof of Theorem 1.1. By assumption,

$$\begin{aligned}\sup_{x\in\mathcal{X}}[\langle\lambda, x\rangle - I(x)] &\geq \sup_{a\in\mathbb{R}}\sup_{x:\langle\lambda,x\rangle>a}[\langle\lambda, x\rangle - I(x)]\\ &\geq \sup_{a\in\mathbb{R}}\left[a - \inf_{x:\langle\lambda,x\rangle>a} I(x)\right]\\ &\geq \sup_{a\in\mathbb{R}}\left[a - \inf_{z>a}\Lambda_\lambda^*(z)\right] = \sup_{z\in\mathbb{R}}[z - \Lambda_\lambda^*(z)].\end{aligned}$$

Note that Λ_λ is convex with $\Lambda_{\lambda(0)} = 0$ and is lower semicontinuous; therefore, it cannot attain the value $-\infty$. By the duality lemma (Lemma D.1), $\sup_{z\in\mathbb{R}}[z - \Lambda_\lambda^*(z)] = \Lambda_\lambda(1) = \Lambda(\lambda)$, proving

$$\sup_{x\in\mathcal{X}}\{\langle\lambda, x\rangle - I(x)\} \geq \Lambda(\lambda).$$

By Lemma 1.3, we have

$$\Lambda(\lambda) \geq \sup_{x\in\mathcal{X}}[\langle\lambda, x\rangle - I(x)].$$

Combining the above results gives $I^* = \Lambda$. Applying the duality lemma (Lemma D.1) again yields $\Lambda^* = I^{**} = I$. □

References

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